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# NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS IN ENGINEERING AND APPLIED SCIENCE

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# DARCY'S LAW FOR FLOW IN POROUS MEDIA AND THE TWO-SPACE METHOD

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## §1. INTRODUCTION

A common problem in engineering and science is to derive simple equations governing complicated phenomena. Often complicated governing equations are known, but they are too difficult to analyze. Therefore it is desired to simplify them in such a way that the phenomena of interest will still be adequately described, while finer details which are not of interest can be disregarded. An example of a simplified equation is Darcy's law, which describes flow of a viscous fluid through a porous medium. The more complicated equation for the same phenomenon is the Navier-Stokes equation. As an example of a general method for simplifying equations, we shall show how to derive Darcy's law from the Navier-Stokes equation.

Simplified equations are often called "homogenized equations," and the procedure of replacing the original equations by them is often called "homogenization." Traditionally, engineers and scientists invent homogenized equations from physical considerations. In recent years, mathematicians have developed new methods for deriving them from the original complicated equations. These methods all involve averaging of one kind or another. In some case it is stochastic averaging, in other cases it is spatial averaging, and sometimes it is space-time averaging.

The introduction of averaging depends upon the fact that the problems involve two widely different scales. There is a long scale of variation which is of interest and is to be retained, and there is a short scale of variation which is not of interest and is to be eliminated. The purpose of averaging is to eliminate the variations on the short scales. Therefore the region over which a spatial average is taken must be large compared to the small scale and small compared to the large scale.

These competing requirements have traditionally made the derivation of simplified equations a heuristic, nonrigorous procedure. However, by the introduction of the two-space method, it is possible to convert it into a formal and rigorous method. In fact, whenever apparently contradictory assumptions are used in a heuristic analysis, that is an indication that the analysis concerns an asymptotic phenomenon. Usually by appropriately recognizing this feature, it is possible to replace the heuristic argument by a systematic and formal asymptotic analysis.

The mathematical study of such asymptotic phenomena has been pursued by Spagnolo [1], DiGiorgi and Spagnolo [2], Sanchez-Palencia [3,4], Babuska [5], Bensoussan, Lions, and Papanicolaou [6,7], Larsen [8,9], McConnell [10], Keller [11], and others. I shall employ the two-space method which I developed for this purpose in 1973. It has been used and improved upon by Bensoussan, Lions, and Papanicolaou [6,7] and Larsen [8,9]. This method has been used by Burridge and Keller [12] to derive the equations of linear poroelasticity from the linearized Navier-Stokes equation and the linearized equations of elasticity. That derivation has many points of similarity to the present one.

Before deriving Darcy's law, we shall consider first a simpler example, that of heat conduction in one dimension. The thermal conductivity will be assumed to vary rapidly with position, and we will seek a less rapidly varying effective conductivity. In Sec. 2 we will derive it from the explicit solution of the heat conduction problem. Then in Sec. 3 we will rederive it by the two-space method, which we shall later use to get Darcy's law.

## §2. EFFECTIVE THERMAL CONDUCTIVITY IN ONE DIMENSION

Let us consider one-dimensional heat flow along a rod in which the thermal conductivity k varies very rapidly with distance x along the rod. To describe rapid variation we write $k = k(x/\varepsilon)$ where $\varepsilon$ is a small parameter. Then $dk/dx = \varepsilon^{-1}k'$, so $dk/dx$ is large when $\varepsilon$ is small even though $k'$ is bounded. If the thermal conductivity also varies on a slow scale, we write $k = k(x,x/\varepsilon)$. Then the steady state temperature distribution along the rod satisfies the equation

$$\frac{d}{dx}\left[k(x,x/\varepsilon)\ \frac{d}{dx}\ u\right] = \frac{d}{dx}\ g(x) \qquad 0 \le x \le 1 \tag{2.1}$$

Here $dg/dx$ represents a heat source, which has been written as a derivative just for convenience, and we also assume that $g(1) = 0$. As boundary conditions we will suppose that one end of the rod is kept at temperature zero and the other end is insulated, so we have

$$u(0) = 0 \qquad \frac{du(1)}{dx} = 0 \tag{2.2}$$

The solution of (2.1) and (2.2) can be found at once by integration. By using the assumption that $g(1) = 0$, we get the result

$$u(x,\varepsilon) = \int_0^x \frac{g(x')}{k(x',x'/\varepsilon)}\ dx' \tag{2.3}$$

We have indicated that u depends upon $\varepsilon$, which is a measure of the small scale on which k varies. As a consequence of the small scale variation of k, u also varies on this small scale. Our goal is to eliminate this small scale variation of u, if possible, by considering the limit of u as $\varepsilon$ tends to zero.

We shall show that under suitable conditions on k, $u(x,\varepsilon)$ has a limit $u_0(x)$,

$$u_0(x) = \lim_{\varepsilon\to 0} u(x,\varepsilon) \tag{2.4}$$

Furthermore, $u_0$ is given explicitly by

$$u_0(x) = \int_0^x \frac{g(x')}{k_0(x')}\ dx' \tag{2.5}$$

Here the "effective conductivity" $k_0(x)$ is defined by

$$\frac{1}{k_0(x)} = \lim_{\varepsilon\to 0} \frac{1}{\Delta x}\int_x^{x+\Delta x} \frac{dx'}{k(x,x'/\varepsilon)}$$
$$= \lim_{\varepsilon\to 0} \frac{\varepsilon}{\Delta x}\int_{x/\varepsilon}^{(x+\Delta x)/\varepsilon} \frac{dy}{k(x,y)} \tag{2.6}$$

Our hypothesis about k is that the limit in (2.6) exists uniformly in x and is independent of $\Delta x$.

Before proving (2.5) we deduce from it that $u_0(x)$ satisfies the equation

$$\frac{d}{dx}\left[k_0(x)\ \frac{d}{dx}\,u_0(x)\right] = \frac{d}{dx}\,g(x) \qquad 0 \le x \le 1 \tag{2.7}$$

and the boundary conditions

$$u_0(0) = 0 \qquad \frac{du_0(1)}{dx} = 0 \tag{2.8}$$

Thus $u_0(x)$ satisfies the same equations as does $u(x)$ with $k(x,x/\varepsilon)$ replaced by $k_0(x)$. This justifies the name "effective conductivity" for $k_0(x)$. We note that $k_0(x)$ is the harmonic mean of $k(x,x/\varepsilon)$ with respect to its second argument.

To prove (2.5) we consider the general integral $I(x,\varepsilon)$ defined by

$$I(x,\varepsilon) = \int_0^x f(x',x'/\varepsilon)\,dx' \tag{2.9}$$

We let $x_j = jx/N$, $j = 0, \ldots, N-1$ where $N > 1$ is an integer, and we set $\Delta x = x/N$. Then we rewrite (2.9) as a sum

$$I(x,\varepsilon) = \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} f(x',x'/\varepsilon)\,dx' \tag{2.10}$$

Next we assume that f is continuously differentiable with respect to its first argument and that $|f_x| \le B$ uniformly in $\varepsilon$. Then we have

$$|f(x',x'/\varepsilon) - f(x_j,x'/\varepsilon)| = |(x' - x_j)f_x(\tilde{x}_j,x'/\varepsilon)| \le B\Delta x$$
$$x_j \le x' \le x_{j+1} \tag{2.11}$$

Here $\tilde{x}_j$ is some point in the interval $x_j \le \tilde{x}_j \le x_{j+1}$.

Now we can use (2.10) and (2.11) to write

$$I(x,\varepsilon) - \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} f(x_j, x'/\varepsilon)dx'$$

$$\le \sum_{j=1}^{N-1} \int_{x_j}^{x_{j+1}} |f(x',x'/\varepsilon) - f(x_j,x'/\varepsilon)|dx'$$

$$\le NB(\Delta x)^2 = Bx^2/N \tag{2.12}$$

Upon taking the limit as $\varepsilon$ tends to zero in (2.12), we obtain

$$\lim_{\varepsilon\to 0} I(x,\varepsilon) - \sum_{j=1}^{N-1} \Delta x \lim_{\varepsilon\to 0} \frac{\varepsilon}{\Delta x} \int_{x_j/\varepsilon}^{(x_j+\Delta x)/\varepsilon} f(x_j,y)dy$$

$$= \lim_{\varepsilon\to 0} I(x,\varepsilon) - \sum_{j=1}^{N-1} \Delta x\, \bar{f}(x_j) \le Bx^2/N \tag{2.13}$$

Here $\bar{f}(x)$ is defined by the following limit, which is assumed to exist uniformly in x and to be independent of $\Delta x$:

$$\bar{f}(x) = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\Delta x} \int_{x/\varepsilon}^{(x+\Delta x)/\varepsilon} f(x,y)dy \tag{2.14}$$

Finally, we let $N \to \infty$ in (2.13), and the last sum tends to the integral of $\bar{f}(x)$ while $Bx^2/N$ tends to zero. Thus we obtain

$$\lim_{\varepsilon\to 0} \int_0^x f(x',x'/\varepsilon)dx' = \int_0^x \bar{f}(x')dx' \tag{2.15}$$

This is the desired result, which we shall summarize as a theorem.

<u>Theorem</u> Suppose that $f_x$ is continuous, that $|f_x(x,x/\varepsilon)| \le B$ uniformly in $\varepsilon$, and that the limit in (2.14) exists uniformly in x and is independent of $\Delta x$. Then (2.15) holds.

The result (2.15) might be called an integration by parts formula, were that name not preempted, since part of the integration is done first in (2.14) and the other part later in (2.15). When the theorem is applied to (2.3) it yields (2.5).

From the method of derivation of the theorem, it is clear how to extend it to the case of more than one rapidly varying argument. The result is

$$\lim_{\varepsilon\to 0} \int_0^x f(x',x'/\varepsilon,x'/\varepsilon^2,\ldots,x'/\varepsilon^n)dx'$$
$$= \int_0^x A_1A_2\cdots A_n f(x',y_1,y_2,\ldots,y_n)dx' \qquad (2.16)$$

Here $A_j$ denotes the operation of averaging with respect to the variable $y_j$. This extended theorem can be applied to (2.3) in which $k = k(x,x/\varepsilon,\ldots,x/\varepsilon^n)$. The result is again (2.5) with $k_0$ given by

$$\frac{1}{k_0(x)} = A_1\cdots A_n \frac{1}{k(x,y_1,\ldots,y_n)} \qquad (2.17)$$

## §3. THE TWO-SPACE METHOD

We shall now solve the problem of the preceding section by the two-space method, which is applicable to more general problems. In doing so we shall permit the source term to be rapidly varying also. Thus we replace (2.1) by

$$\partial_x[k(x,x/\varepsilon)\partial_x]u = h(x,x/\varepsilon) \qquad 0 \le x \le 1 \qquad (3.1)$$

To solve this equation we define $y = x/\varepsilon$, write $u(x,\varepsilon) = v(x,y,\varepsilon)$ and replace $\partial_x$ by $\partial_x + \varepsilon^{-1}\partial_y$ in (3.1). Then (3.1) becomes

$$[\varepsilon^{-2}\partial_y k(x,y)\partial_y + \varepsilon^{-1}(\partial_x k\partial_y + \partial_y k\partial_x) + \partial_x k\partial_x]v = h(x,y) \qquad (3.2)$$

Now we seek $v$ in the form

$$v(x,y,\varepsilon) = v_0(x,y) + \varepsilon v_1(x,y) + \varepsilon^2 v_2(x,y) + o(\varepsilon^2) \qquad (3.3)$$

We substitute (3.3) into (3.2) and equate coefficients of the first three powers of $\varepsilon$ to obtain

$$\partial_y k\partial_y v_0 = 0 \qquad (3.4)$$

$$\partial_y k\partial_y v_1 = -(\partial_x k\partial_y + \partial_y k\partial_x)v_0 \qquad (3.5)$$

$$\partial_y k\partial_y v_2 = -(\partial_x k\partial_y + \partial_y k\partial_x)v_1 - \partial_x k\partial_x v_0 + h \qquad (3.6)$$

The solution of (3.4) with $y_0$ arbitrary is

$$v_0(x,y) = v_0(x,y_0) + k(x,y_0)[\partial_y v_0(x,y_0)] \int_{y_0}^{y} k^{-1}(x,y')dy' \tag{3.7}$$

We now require that $v_0(x,y)$ be a bounded function of y. Then if k is positive and bounded, as we assume, $v_0$ given by (3.7) will not be bounded unless $\partial_y v_0(x,y_0) = 0$. Since $y_0$ is arbitrary this implies that $v_0$ is independent of y:

$$v_0 = v_0(x) \tag{3.8}$$

Upon using (3.8) in (3.5) and solving for $v_1$, we get

$$v_1(x,y) = v_1(x,y_0) - (y - y_0)\partial_x v_0(x) + k(x,y_0)[\partial_y v_1(x,y_0) + \partial_x v_0(x)] \int_{y_0}^{y} k^{-1}(x,y')dy' \tag{3.9}$$

We next impose the requirement that $v_1$ be a bounded function of y. Then upon dividing (3.9) by $y - y_0$ and letting $y \to \infty$, we get

$$\partial_x v_0(x) = k(x,y_0)[\partial_y v_1(x,y_0) + \partial_x v_0(x)]k_0^{-1}(x) \tag{3.10}$$

Here $k_0(x)$ is defined by the following limit, which we assume to exist and to be independent of $y_0$:

$$k_0^{-1}(x) = \lim_{y\to\infty} \frac{1}{y - y_0} \int_{y_0}^{y} k^{-1}(x,y')dy' \tag{3.11}$$

Now we solve (3.10) for $k\partial_y v_1$ and substitute the result into (3.6), which we rewrite as

$$\partial_y k\partial_y v_2 + \partial_y k\partial_x v_1 = -\partial_x[k_0(x)\partial_x v_0(x)] + h(x,y) \tag{3.12}$$

Integration of (3.12) from $y_0$ to y yields

$$(k\partial_y v_2 + k\partial_x v_1)\Big|_{y_0}^{y} = -(y - y_0)\partial_x[k_0(x)\partial_x v_0(x)] + \int_{y_0}^{y} h(x,y')dy' \tag{3.13}$$

Dividing (3.13) by $y - y_0$ and letting $y \to \infty$ yields, in view of the assumed boundedness of terms on the left side of (3.13),

$$\partial_x[k_0(x)\partial_x v_0(x)] = \bar{h}(x) \tag{3.14}$$

Here $\bar{h}(x)$ is the average of $h(x,y)$ with respect to $y$.

The result (3.14) is the desired equation for $v_0(x)$, the leading term in $u(x,\varepsilon) = v(x,y,\varepsilon)$. The effective conductivity $k_0(x)$ given by (3.11) is the same as that defined by (2.6) in Sec. 2.

## §4. FLOW IN POROUS MEDIA

Let us now consider the flow of a compressible viscous fluid through a rigid porous medium. The pore "diameter" is assumed to be small compared to the macroscopic scale of the medium. We shall denote by $\varepsilon$ the ratio of the pore diameter to the macroscopic scale, and by $D_\varepsilon$ the interior of all the pores. Then the equations governing the fluid flow, which are the Navier-Stokes equation, the continuity equation, and the equation of state, hold in $D_\varepsilon$. These equations are

$$\rho(\partial_t u + u\cdot\nabla u) = -\nabla p + \mu(\nabla^2 + \frac{1}{3}\nabla\nabla\cdot)u + f \tag{4.1}$$

$$\partial_t \rho + \nabla\cdot(\rho u) = 0 \tag{4.2}$$

$$\rho = \rho(p) \tag{4.3}$$

Here $\rho$, $p$, $u$, and $\mu$ are the fluid density, pressure, velocity, and viscosity coefficient, respectively, while $f$ is the external body force per unit volume. In addition to these equations, we have $u = 0$ on $\partial D_\varepsilon$.

Our goal is to derive simplified equations for the fluid motion by taking advantage of the smallness of $\varepsilon$. To this end we first introduce $y$ and $\tau$ defined by

$$y = x/\varepsilon \qquad \tau = t/\varepsilon^{1/2} \tag{4.4}$$

Then we write $u$, $\rho$, $p$, $\mu$, and $f$ in the forms

$$u = \varepsilon^{1/2}\tilde{u}(x,y,\tau,\varepsilon) \qquad \rho = \tilde{\rho}(x,y,\tau,\varepsilon) \qquad p = \tilde{p}(x,y,\tau,\varepsilon)$$
$$f = \tilde{f}(x,y,\tau,\varepsilon) \qquad \mu = \varepsilon^{3/2}\tilde{\mu} \tag{4.5}$$

Note that $\mu$ is required to be small of order $\varepsilon^{3/2}$. We use these variables in (4.1)-(4.3) and replace $\nabla$ by $\nabla_x + \varepsilon^{-1}\nabla_y$ to obtain

$$\tilde{\rho}[\partial_\tau \tilde{u} + \tilde{u}\cdot(\nabla_y + \varepsilon\nabla_x)\tilde{u}] = -(\varepsilon^{-1}\nabla_y + \nabla_x)\tilde{p} + \tilde{\mu}[(\nabla_y + \varepsilon\nabla_x)^2 + \frac{1}{3}(\nabla_y + \varepsilon\nabla_x)(\nabla_y + \varepsilon\nabla_x)\cdot]\tilde{u} + \tilde{f} \tag{4.6}$$

$$\partial_\tau\tilde{\rho} + (\nabla_y + \varepsilon\nabla_x)\cdot(\tilde{\rho}\tilde{u}) = 0 \tag{4.7}$$

$$\tilde{\rho} = \rho(\tilde{p}) \tag{4.8}$$

To solve these equations, we assume that $\tilde{u}$, $\tilde{\rho}$, $\tilde{p}$, and $\tilde{f}$ have expansions of the form

$$\begin{aligned}
\tilde{u}(x,y,\tau,\varepsilon) &= u_0(x,y,\tau) + \varepsilon u_1(x,y,\tau) + o(\varepsilon)\\
\tilde{\rho} &= \rho_0 + \varepsilon\rho_1 + o(\varepsilon)\\
\tilde{p} &= p_0 + \varepsilon p_1 + o(\varepsilon)\\
\tilde{f} &= f_0(x,\tau) + \varepsilon f_1 + o(\varepsilon)
\end{aligned} \tag{4.9}$$

We substitute (4.9) into (4.6)-(4.8) and equate coefficients of the lowest power of $\varepsilon$ in each equation to get

$$\nabla_y p_0(x,y,\tau) = 0 \tag{4.10}$$

$$\partial_\tau\rho_0 + \nabla_y\cdot(\rho_0 u_0) = 0 \tag{4.11}$$

$$\rho_0 = \rho(p_0) \tag{4.12}$$

Then from the next lowest power of $\varepsilon$ we obtain

$$\rho_0(\partial_\tau u_0 + u_0\cdot\nabla_y u_0) = -\nabla_y p_1 + \tilde{\mu}(\nabla_y^2 + \frac{1}{3}\nabla_y\nabla_y\cdot)u_0 + f_0 - \nabla_x p_0 \tag{4.13}$$

$$\partial_\tau\rho_1 + \nabla_y\cdot(\rho_0 u_1 + \rho_1 u_0) + \nabla_x\cdot(\rho_0 u_0) = 0 \tag{4.14}$$

$$\rho_1 = \rho_p(p_0)p_1 \tag{4.15}$$

From (4.9) we see that $p_0$ is independent of y, and then from (4.12) so is $\rho_0$:

$$p_0 = p_0(x,\tau) \tag{4.16}$$

$$\rho_0 = \rho_0(x,\tau) = \rho[p_0(x,\tau)] \tag{4.17}$$

Next from (4.1) we find that $\nabla_y \cdot u_0 = -\rho_0^{-1}\partial_\tau \rho_0$, and from (4.17) we conclude that $\nabla_y \cdot u_0$ is independent of y. Therefore, $\nabla_y \nabla_y \cdot u_0 = 0$, so this term can be omitted from (4.13).

Now we consider (4.11) and (4.13) as a system of equations for $u_0$ and $p_1$ as functions of y and $\tau$, in which x occurs as a parameter. In these equations we view $\rho_0(x,\tau)$ and $f_0(x,\tau) - \nabla_x p_0(x,\tau)$ as a given time dependent density and a given time dependent body force, respectively. Both of them are independent of y. We seek a solution $u_0$, $p_1$ of these equations with the normal component of $u_0$ vanishing on the boundary of the domain, which may depend upon x. If the medium is spatially *periodic in y*, we require the solution to have the same periodicity; otherwise, we require it to be a bounded function of y. In either case we assume that the solution is unique. Then we can write it as a functional of $\rho_0$ and $f_0 - \nabla_x p_0$ in the form

$$u_0(x,y,\tau) = U[x,y,\tau,\rho_0,f_0 - \nabla_x p_0] \tag{4.18}$$

$$p_1(x,y,\tau) = P[x,y,\tau,\rho_0,f_0 - \nabla_x p_0] \tag{4.19}$$

The functionals U and P involve their argument functions only for $\tau' \leq \tau$.

We now define the average of any function of y by integrating it with respect to y over a large domain D of the fluid, dividing the integral by the volume V of D, and letting D and V tend to infinity. In particular, by averaging (4.18) and (4.19) we obtain

$$\begin{aligned}\bar{u}_0(x,\tau) &= \bar{U}[x,\tau,\rho_0,f_0 - \nabla_x p_0] \\ &= \lim_{V\to\infty} \frac{1}{V}\int_D U[x,y,\tau,\rho_0,f_0 - \nabla_x p_0]\,dy \end{aligned}\tag{4.20}$$

$$\begin{aligned}\bar{p}_1(x,\tau) &= \bar{P}[x,\tau,\rho_0,f_0 - \nabla_x p_0] \\ &= \lim_{V\to\infty} \frac{1}{V}\int_D P[x,y,\tau,\rho_0,f_0 - \nabla_x p_0]\,dy \end{aligned}\tag{4.21}$$

These two relations express $\bar{u}_0$ and $\bar{p}_1$ as functionals of $\nabla_x p_0$ and $\rho_0$, so together they represent a generalization of Darcy's law to nonlinear time dependent compressible flows. We shall examine them further in the next section.

## §5. DARCY'S LAW

Relations (4.20) and (4.21) can be used to derive an equation for $p_0(x,\tau)$. To obtain it we average (4.14) in the manner just described. Integration of the term $\nabla_y \cdot (\rho_0 u_1 + \rho_1 u_0)$ yields a surface integral over the "ends" of the pores, since $u_0$ and $u_1$ vanish on the rigid boundaries. When divided by V this integral tends to zero, and then (4.14) leads to

$$\partial_\tau [\rho_p(p_0)\bar{p}_1] + \nabla_x \cdot [\rho_0 \bar{u}_0] = 0 \tag{5.1}$$

In (5.1) we have used (4.15) to replace $\rho_1$ by $\rho_p p_1$. Now we use (4.20) and (4.21) for $\bar{u}_0$ and $\bar{p}_1$ in (5.1) to obtain

$$\begin{aligned}&\partial_\tau \{\rho_p(p_0)\bar{P}[x,\tau,\rho_0,f_0 - \nabla_x p_0]\} \\ &+ \nabla_x \cdot \{\rho_0 \bar{U}[x,\tau,\rho_0,f_0 - \nabla_x p_0]\} = 0\end{aligned} \tag{5.2}$$

Equations (5.2) and (4.17) are a pair of equations for $p_0(x,t)$ and $\rho_0(x,t)$. If (4.17) is used to eliminate $\rho_0$, the result is a single equation for $p_0(x,\tau)$.

We shall now consider some special cases of (4.20) and (4.21) which are closer to the usual form of Darcy's law. First, for an incompressible fluid of constant density $\rho_0$, we need not indicate the argument $\rho_0$ in (4.20), which then becomes

$$\bar{u}_0(x,\tau) = \bar{U}[x,\tau,f_0 - \nabla_x p_0] \tag{5.3}$$

Furthermore, $\rho_p = 0$ so $\rho_p(p_0)\bar{P} = 0$, and then (5.2) simplifies to the following equation for $p_0(x,\tau)$:

$$\nabla_x \cdot \bar{U}[x,\tau,f_0 - \nabla_x p_0] = 0 \tag{5.4}$$

In the steady incompressible case $\bar{U}$ is just a function of $f_0 - \nabla_x p_0$, rather than a functional, and it is independent of $\tau$. Then (5.3) and (5.4) simplify accordingly, and (5.3) is a nonlinear form of Darcy's law for steady incompressible flow.

In the case of steady compressible flow, (4.20) and (4.21) become

$$\bar{u}_0(x) = \bar{U}[x,\rho_0(x),f_0(x) - \nabla_x p_0(x)] \tag{5.5}$$

$$\bar{p}_1(x) = \bar{P}[x,\rho_0(x),f_0(x) - \nabla_x p_0(x)] \tag{5.6}$$

Here $\bar{U}$ and $\bar{P}$ are ordinary functions, and not functionals, so (5.5) is a nonlinear Darcy law for steady compressible flow. The corresponding equation for $p_0$ is, from (5.2),

$$\nabla_x \cdot \{\rho_0 \bar{U}[x,\rho_0,f_0 - \nabla_x p_0]\} = 0 \tag{5.7}$$

Finally, we shall consider the case in which $\rho_0(x,\tau)$ is close to a constant density m and $f_0 - \nabla_x p_0$ is close to zero. Then we can linearize the functionals $\bar{U}$ and $\bar{P}$. In doing so, it is helpful to return to Eqs. (4.11) and (4.13) by means of which U and P are defined. The linearized forms of these equations are

$$\partial_\tau \rho_0 + m\nabla_y \cdot u_0 = 0 \tag{5.8}$$

$$m\partial_\tau u_0 = -\nabla_y p_1 + \tilde{\mu}(\nabla_y^2 + \frac{1}{3}\nabla_y \nabla_y \cdot)u_0 + f_0 - \nabla_x p_0 \tag{5.9}$$

We seek the bounded solution $u_0$, $p_1$ of these equations which has vanishing velocity on the rigid boundaries of the pores. This solution is a linear functional of the inhomogeneous terms $\partial_\tau \rho_0$ and $f_0 - \nabla_x p_0$. Therefore we write it in the form

$$u_0(x,y,\tau) = A(x,y,\tau)(f_0 - \nabla_x p_0) + B(x,y,\tau)\partial_\tau \rho_0 \tag{5.10}$$

$$p_1(x,y,\tau) = C(x,y,\tau)(f_0 - \nabla_x p_0) + E(x,y,\tau)\partial_\tau \rho_0 \tag{5.11}$$

Then

$$\bar{u}_0(x,\tau) = \bar{A}(x,\tau)(f_0 - \nabla_x p_0) + \bar{B}(x,\tau)\partial_\tau \rho_0 \tag{5.12}$$

$$\bar{p}_1(x,\tau) = \bar{C}(x,\tau)(f_0 - \nabla_x p_0) + \bar{E}(x,\tau)\partial_\tau \rho_0 \tag{5.13}$$

These results are the linearized forms of the generalized Darcy law for time dependent compressible flows.

The linearized form of (5.1), in view of (5.12) and (5.13), is

$$\rho_p(\pi)\partial_\tau[\bar{C}(f_0 - \nabla_x p_0) + \bar{E}\partial_\tau\rho_0]$$
$$+ m\nabla_x\cdot[\bar{A}(f_0 - \nabla_x p_0) + \bar{B}\partial_\tau\rho_0] = 0 \qquad (5.14)$$

Here $\pi$ is the constant unperturbed pressure corresponding to the density m.

In the steady case (5.12) simplifies to the usual form of Darcy's law

$$\bar{u}_0(x) = \bar{A}(x)[f_0(x) - \nabla_x p_0(x)] \qquad (5.15)$$

Here $\bar{A}(x)$ is just a matrix which transforms as a tensor. The steady form of (5.14) is a linear equation for $p_0(x)$:

$$\nabla_x\cdot\{\bar{A}(x)[f_0 - \nabla_x p_0]\} = 0 \qquad (5.16)$$

For an incompressible fluid (5.12) simplifies to

$$\bar{u}_0(x,\tau) = \bar{A}(x,\tau)(f_0 - \nabla_x p_0) \qquad (5.17)$$

Since $\rho_p = 0$ in this case, (5.14) becomes

$$\nabla_x\cdot\{\bar{A}(f_0 - \nabla_x p_0)\} = 0 \qquad (5.18)$$

In the steady incompressible case, (5.17) reduces to the usual Darcy law (5.15). When the medium is macroscopically isotropic, $\bar{A}(x)$ is a scalar.

## §6. CONCLUSION

After illustrating, on one-dimensional heat conduction, the two-space method for deriving simplified equations, we have applied it to flow in porous media. Then we were able to derive Darcy's law and various generalizations of it. To determine the coefficients in this law and its generalizations, it is necessary to solve a difficult problem which we have not considered. That problem involves the detailed pore configuration. Our goal was to determine the form of the law and to characterize the coefficients in terms of the solution of a specific problem. In the heat conduction problem we could actually find the coefficient because the problem was so simple.

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8